Available online at www.sciencedirect.com
SCIENCE DIRECT®
International Journal of Fatigue 27 (2005) 954–965
International Journal of Fatigue
www.elsevier.com/locate/ijfatigue

Fatigue life prediction based on variable amplitude tests—methodology

Pär Johannesson[a,*], Thomas Svensson[a], Jacques de Maré[a,b]

[a]Fraunhofer-Chalmers Centre, Chalmers Science Park, 412 88 Göteborg, Sweden
[b]Mathematical Statistics, Chalmers University of Technology, 412 96 Göteborg, Sweden

Received 21 July 2004; received in revised form 31 August 2004; accepted 18 November 2004

Abstract

The reliability of mechanical systems depends in part on the fatigue strength of the material, whose properties need to be determined by experiments. In a standard fatigue test a Wöhler curve is established, i.e. the relation between the life in the number of cycles and the load amplitude. Based on a damage accumulation rule, the life can be predicted for service loads. However, most service loads are far from constant amplitude loads, often resulting in large systematic errors in life predictions. The method in this paper is based on an estimation of the Wöhler curve directly from variable amplitude tests. Consequently, both the life prediction and estimation are performed at similar load situations. In the Wöhler curve the constant amplitude is replaced by an equivalent amplitude given by the load spectrum. An important part of this paper is the analysis of the statistical uncertainties in life estimation and prediction. The features and usefulness of our method are demonstrated by data from Agerskov. Our method can be seen as an extension of the Gassner line and the relative Miner rule.
© 2005 Elsevier Ltd. All rights reserved.

Keywords: Life prediction; Load spectrum; Service loads; Wöhler curve; Palmgren-Miner hypothesis

1. Introduction

Traditionally fatigue life at variable amplitude is predicted by using material properties from constant amplitude laboratory tests together with the Palmgren-Miner [13,9] damage accumulation hypothesis. The tests are carried out for a number of different load amplitudes, keeping a fixed load ratio $R=S_{\min}/S_{\max}$, and a few replicates are made at each amplitude. The resulting number of cycles to failure N is plotted versus the load amplitude S, the so-called Wöhler diagram, and the material parameters α and β are estimated from the Basquin relation [3],

$$N=\alpha S^{-\beta}. \tag{1}$$

The resulting estimates $\hat{\alpha}$ and $\hat{\beta}$ are regarded as a characterisation of the fatigue property of the material in question at the specified R-ratio.

The estimated parameters are used to predict the fatigue life for the material at a variable amplitude load together with the Palmgren-Miner hypothesis of cumulative damage,

$$D=\sum_i \frac{n_i}{N_i}=\sum_i \frac{n_i}{\alpha S_i^{-\beta}}\approx\sum_i \frac{n_i}{\hat{\alpha} S_i^{-\hat{\beta}}}=\frac{1}{\hat{\alpha}}\sum_i n_i S_{i\hat{\beta}} \tag{2}$$

where failure is predicted as the number of cycles where the damage sum D equals one. Here, n_i is the number of cycles of amplitude S_i that are obtained by a cycle counting procedure, using for instance the Rain-Flow-Count algorithm. One complication is that the load ratio R may be different for the different cycles in the spectrum, and the parameters are only valid for the R-value used in the constant amplitude test. This can be addressed by using an empirical correction formula, or by using the global R-value for the variable amplitude load process, or simply by neglecting the mean stress effect.

Validation tests have shown that this procedure often gives poor predictions and the use of the Palmgren-Miner hypothesis has been questioned. An example of where the predictions are non-conservative is shown in Fig. 1, where the data comes from Agerskov [2]. However, alternative methods need a lot of computational effort making cycle by cycle calculations, and require detailed knowledge about

* Corresponding author. Tel.: +46 31 772 4295; fax: +46 31 772 4260.
E-mail addresses: par.johannesson@fcc.chalmers.se (P. Johannesson), thomas.svensson@fcc.chalmers.se (T. Svensson), demare@math.chalmers.se (J. de Maré).

0142-1123/$ - see front matter © 2005 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ijfatigue.2004.11.009

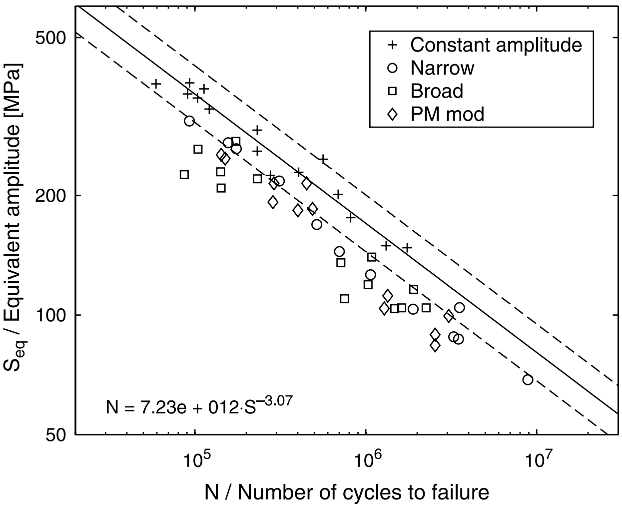

Fig. 1. *SN*-curve estimated from constant amplitude tests, with scatter bands of two standard deviations. The lives for the Broad, Narrow, and PM mod spectra tests are plotted versus the equivalent loads that are calculated using the estimated β.

the load history, crack geometry, crack growth laws, and other mechanisms. In many applications this information is not fully available, and simpler methods must be used.

Some reasons for the lack of validity of the Palmgren-Miner rule that have been identified are:

(1) sequential effects, i.e. the order of cycle application to the material is essential,
(2) residual stresses which may relax at variable amplitude, but remain at constant amplitude,
(3) threshold effects are different when small cycles are mixed up with large ones.

These effects may be interpreted as model errors which are the result of the extrapolation from an extreme load situation (constant amplitude) to the service load situation with a mixture of different load cycles. One possible way to diminish the influence of these effects is to perform the laboratory material tests at a situation more close to the service, but still use the simple Palmgren-Miner hypothesis.

This idea has been evaluated in different ways, e.g. the Gassner line [6], where the maximum load is ploted versus the life, and the relative Miner rule [18,16] that modifies the damage value at failure. In this paper we propose a general formulation of the idea and compare it with the similar methods, which can all be seen as special cases of the proposed method.

The proposed method for life prediction based on variable amplitude tests is presented in Sections 2 and 3, where the statistical details are found in Appendix A. In Section 4 the methodology is applied to a data set from literature. Similar proposals are reviewed in Section 5, and possible extensions of our method are discussed in Section 6, which is followed by the conclusions in Section 7.

2. Proposed method for life prediction

Our proposal is to perform laboratory reference tests, not with constant amplitude, but with one or more reference load spectra scaled to different levels. For each individual spectrum we define an equivalent load amplitude (or equivalent load range)

$$S_{\text{eq}} = \sqrt[\beta]{\sum_k \nu_k S_k^\beta}. \tag{3}$$

where β is the exponent in the Basquin Eq. (1) and ν_k is the relative frequency of occurrence of the load amplitude S_k in the spectrum. Often tests are performed by repeating a load block, with n_k cycles of amplitude S_k, say. In this case the relative frequency ν_k is defined as n_k over the number of cycles in the block, and the number of cycles to failure is defined as the number of cycles in the block multiplied by the number of repetitions of the block. The shape and the scaling of the load spectra are chosen to give different equivalent load amplitudes, and the material parameters α and β are estimated from the following variant of Eq. (1)

$$N = \alpha S_{\text{eq}}^{-\beta}. \tag{4}$$

The estimation is not as easy here as in the traditional case, since the parameter β appears not only in Eq. (4), but also in the equivalent load through Eq. (3). However, it can be solved by for instance the Maximum-Likelihood method combined with numerical optimisation.

Predictions for another spectrum are obtained using

$$\hat{N} = \hat{\alpha}\hat{S}_{\text{eq}}^{-\hat{\beta}} \quad \text{where} \quad \hat{S}_{\text{eq}} = \sqrt[\hat{\beta}]{\sum_k \hat{\nu}_k \hat{S}_k^{\hat{\beta}}} \tag{5}$$

and $\{\hat{\nu}_k, \hat{S}_k; k=1,2,\ldots\}$ is the spectrum of the component whose life we want to predict. The advantage of this proposal compared to the traditional method is that the reference tests can be made with load spectra closer to the ones that need to be predicted. Consequently, the above mentioned model errors are expected to be reduced, and the Palmgren-Miner rule should agree better.

3. Estimating the *SN*-curve from variable amplitude tests

To model the scatter in observed lives, an multiplicative error is introduced in the *SN*-curve of Eq. (4), giving for the ith test with spectrum $\{\nu_{i,k},S_{i,k};k=1,2,\ldots\}$

$$N_i = \alpha S_{\text{eq},i}^{-\beta}\epsilon_i, \quad i = 1,\ldots,n \tag{6}$$

where n is the number of tests, and ϵ_i is modelled as independent log-normally distributed errors. For the statistical analysis it is convenient to rewrite the model Eq. (6) by taking logs and subtracting the mean of $\ln S_{\text{eq},i}$, arriving at the non-linear regression model

$$y_i = a - \beta(\ln S_{\mathrm{eq},i} - \overline{\ln S_{\mathrm{eq}}}) + e_i = m_i(a,\beta) + e_i,$$
$$\overline{\ln S_{\mathrm{eq}}} = \frac{1}{n}\sum_{j=1}^{n} \ln S_{\mathrm{eq},j} \tag{7}$$

where $y_i = \ln N_i$, and $e_i = \ln \epsilon_i$ is $N(0, \sigma^2)$. The parameter α of the *SN*-curve is thus obtained as

$$\alpha = \exp(a + \beta\overline{\ln S_{\mathrm{eq}}}). \tag{8}$$

The estimates of the parameters a and β are defined using the maximum likelihood method. Here we will only present the results, but full details on the derivation is found in Appendix A. It turns out that the maximum likelihood estimates $\hat{a}$ and $\hat{\beta}$ coincides with the least-squares estimates. The estimate of a is the mean of the log lives

$$\hat{a} = \frac{1}{n}\sum_{i=1}^{n} y_i \tag{9}$$

and the estimate $\hat{\beta}$ of β is found by minimising the function

$$g(\beta) = \sum_{i=1}^{n} (y_i - m_i(\hat{a}, \beta))^2. \tag{10}$$

The estimate of σ^2 is

$$s^2 = \frac{1}{n-2}\sum_{i=1}^{n} (y_i - m_i(\hat{a}, \hat{\beta}))^2. \tag{11}$$

The estimates are (at least asymptotically) independent and distributed as

$$\hat{a} \in N(a, \sigma^2/n), \hat{\beta} \in N(\beta, \sigma^2/q), (n-2)s^2/\sigma^2 \in \chi^2(n-2) \tag{12}$$

with

$$q = \sum_{i=1}^{n} (c_i - \bar{c})^2, \quad c_i = \frac{\sum_k \nu_{i,k} S_{i,k}^{\beta} \ln S_{i,k}}{\sum_k \nu_{i,k} S_{i,k}^{\beta}}, \quad \bar{c} = \frac{1}{n}\sum_{i=1}^{n} c_i. \tag{13}$$

The $1-p$ confidence intervals for the parameters a, β, and σ are then

$$I_a = (\hat{a} - t_{1-p/2}(n-2)\cdot s/\sqrt{n}, \quad \hat{a} + t_{1-p/2}(n-2)\cdot s/\sqrt{n}), \tag{14}$$

$$I_\beta = (\hat{\beta} - t_{1-p/2}(n-2)\cdot s/\sqrt{q}, \quad \hat{\beta} + t_{1-p/2}(n-2)\cdot s/\sqrt{q}), \tag{15}$$

$$I_\sigma = \left(s\sqrt{\frac{n-2}{\chi^2_{1-p/2}(n-2)}}, \quad s\sqrt{\frac{n-2}{\chi^2_{p/2}(n-2)}} \right) \tag{16}$$

where $t_{1-p/2}(n-2)$ is the $1-p/2$ quantile of the Student's t-distribution, and $\chi^2_{1-p/2}(n-2)$ is the $1-p/2$ quantile of the χ^2-distribution, both with $n-2$ degrees of freedom.

It should be noted that a constant amplitude load is in fact a degenerate case of a variable amplitude load, with only one amplitude, say S, and consequently the equivalent amplitude becomes $S_{\mathrm{eq}} = S$, and $c_i = \ln S$. This transforms the derived formulas (6)–(16) exactly into those that are known for the constant amplitude case.

3.1. Life prediction and its uncertainty

The life prediction for a new load spectrum $\{\hat{\nu}_k, \hat{S}_k; k = 1, 2, \ldots\}$ is calculated as

$$\hat{N} = \hat{\alpha}\hat{S}_{\mathrm{eq}}^{-\hat{\beta}}. \tag{17}$$

The $1-p$ confidence interval for the median life is

$$I_{\hat{N}} = \left(\exp\left(\ln \hat{N} \pm t_{1-p/2}(n-2)\cdot s\sqrt{\frac{1}{n} + \frac{(\hat{c}-\bar{c})^2}{q}} \right) \right) \tag{18}$$

and the $1-p$ prediction interval for a new fatigue test is

$$I\tilde{N} = \left(\exp\left(\ln\hat{N} \pm t_{1-p/2}(n-2)\cdot s\sqrt{\frac{1}{n} + \frac{(\hat{c}-\bar{c})^2}{q} + 1} \right) \right) \tag{19}$$

with

$$\hat{c} = \frac{\sum_k \hat{\nu}_k \hat{S}_k^{\hat{\beta}} \ln \hat{S}_k}{\sum_k \hat{\nu}_k \hat{S}_k^{\hat{\beta}}}. \tag{20}$$

4. The Agerskov data

Our suggested methodology will be studied using data from literature, see Agerskov and Ibsø [2] and Agerskov [1]. They tested welded steel plate specimens with longitudinal attachments under both constant amplitude loading and variable amplitude loads. Three offshore load spectra were used, *Broad*, a broad band spectrum, *Narrow*, a narrow band spectrum, PM *mod*, a modified Pierson-Moscowitz wave elevation spectrum with broad band character.

4.1. Life prediction based on constant amplitude tests

The results from *SN*-tests are often presented only as the estimated curve, and sometimes also the scatter bands are shown, like in Fig. 1 for constant amplitude, where the ± 2 standard deviations scatter bands are given. The interpretation of the scatter bands can be based on the normal distribution, which translates them into the statement that, in the long run, 95% of the future tests should fall within the scatter bands. However, the error of the former statement is that our lack of knowledge about the true *SN*-curve is not

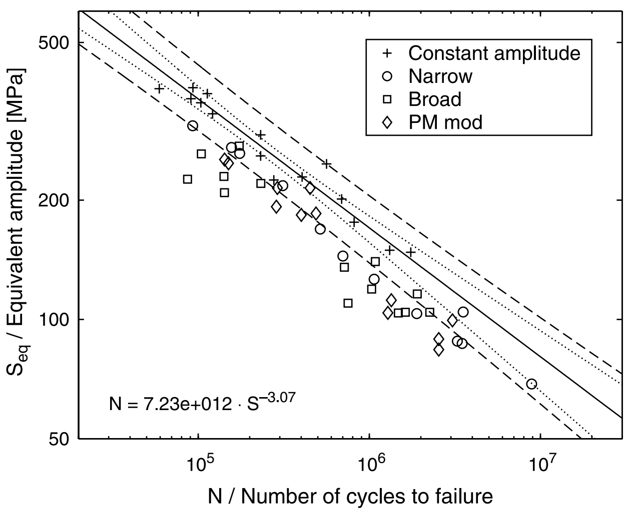

Fig. 2. *SN*-curve estimated from constant amplitude tests, with prediction and confidence bands for constant amplitude loads. The lives for the Broad, Narrow, and PM mod spectra tests are plotted versus the equivalent loads calculated using the estimated β.

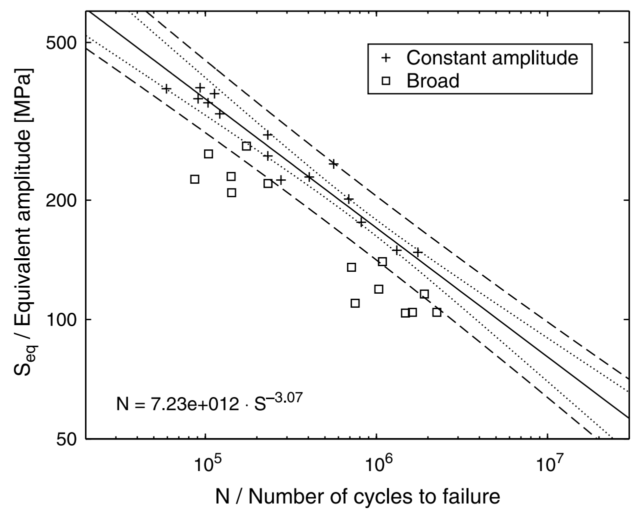

Fig. 3. *SN*-curve estimated from constant amplitude tests, with prediction and confidence bands for Broad spectrum.

included in the scatter bands, only the uncertainty around the true *SN*-curve is included. At the midpoint of our observations the scatter bands will often give a good approximation, but when we move further out on the *SN*-curve the uncertainty in the predicted life can be severely underestimated if it is based on scatter bands. The correct prediction intervals (see Eq. (19) and Fig. 2) take into account the intrinsic uncertainty around the *SN*-curve, as well as the uncertainty in the estimated *SN*-curve. The difference between the scatter bands and the prediction intervals is seen by comparing Figs. 1 and 2.

The observed lives from the variable amplitude tests are shown in Figs. 1 and 2, all being non-conservatively predicted. Next we will discuss this systematic prediction error, and prediction for spectrum loads. The confidence and prediction intervals for variable amplitude depend on the load spectrum. The reason for this is that the equivalent amplitude depends on both the load spectrum and the damage exponent that were estimated. Consequently, there is an uncertainty in S_{eq} which has to be taken into account. The prediction intervals for the Broad spectrum are presented in Fig. 3, where we observe only a minor change compared to Fig. 2.

From Fig. 3 we would like to draw the conclusion that for this specimen, an *SN*-curve based on CA gives non-conservative life predictions, since all observations from Broad are below the estimated *SN*-curve, and about half of the observations fall outside their prediction intervals, when only about 5% should. However, this observed systematic error in the predicted life could be an effect of the uncertainty in the estimated *SN*-curve. To make a more precise statement on the presence of a systematic error, we can either compare the estimates from CA with estimates based on variable amplitude tests, or we can examine the mean of the prediction errors. The former will be discussed later. Now we will examine the relative error in life prediction, $N_{\mathrm{rel}}=N/N_{\mathrm{pred}}$, which is estimated as the geometric mean of the observed lives over the predicted lives, $\tilde{N}_j/\hat{N}_j$, see Eqs. (A.64)–(A.65) in Appendix A. In the relative Miner rule it is used as the Miner sum at failure, which is the same as adjusting the parameter α. The relative prediction error is estimated at 0.38, and a 95% confidence interval is $0.31<N/N_{\mathrm{pred}}<0.47$, which shows a statistically significant difference from one, and consequntly there exists a systematic error in the life prediction for Broad, when the *SN*-curve is based on constant amplitude tests.

4.2. Life prediction based on variable amplitude tests

Our method aims at avoiding the systematic life prediction error by estimating the *SN*-curve directly from variable amplitude tests. Based on the test results with Broad spectrum, the *SN*-curve in Fig. 4 has been obtained, which gives a good fit to the observed lives. However, when using this *SN*-curve, the life predictions for constant amplitude is conservative by a factor 2.57, with 95% confidence interval of $1.82<N/N_{\mathrm{pred}}<3.63$. Again, the systematic error in life prediction between Broad and CA is observed.

The predictions for the Narrow spectrum using the *SN*-curve based on the Broad spectrum tests is shown in Fig. 5. The relative life prediction error is estimated at 1.42, with a 95% confidence interval $0.98<N/N_{\mathrm{pred}}<2.05$, which covers unity. The goal was that the systematic model errors would diminish when using variable amplitude loads both for estimation and prediction. Even though the spectra here have quite different shapes, no systematic error in the life predictions can be statistically ensured. If the life predictions for the Narrow spectrum is based on constant amplitude tests, the life predictions will be underestimated

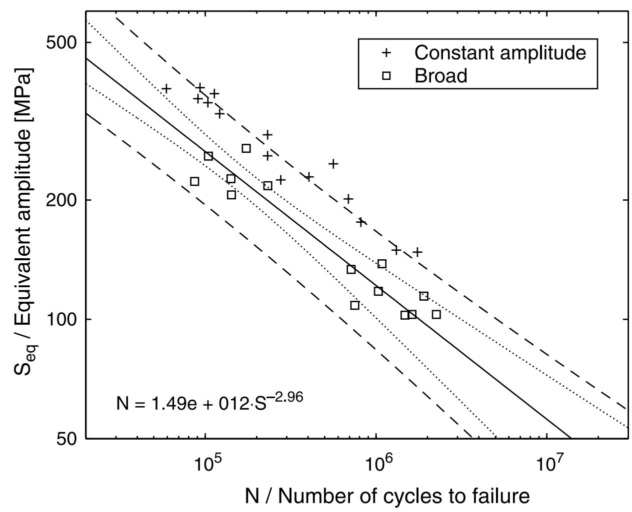

Fig. 4. *SN*-curve estimated from Broad spectrum tests, with prediction and confidence bands for constant amplitude.

by a factor 0.53 (95% interval $0.41 < N/N_{\mathrm{pred}} < 0.69$), showing a statistically significant error.

An important feature of our method is that many different types of spectra can be used in the estimation of the *SN*-curve. In our present example the test results from Broad, Narrow, and PM-mod are combined, resulting in the *SN*-curve presented in Fig. 6. The constant amplitude tests are off by a factor 2.11 (95% confidence interval $1.69 < N/N_{\mathrm{pred}} < 2.64$).

4.3. Comparison of SN-curves based on CA and VA tests

Next we will concentrate on comparing the estimates from constant and variable amplitude tests, in order to examine which parameters are different.

The first thing is to examine the estimates of the parameters of the *SN*-curve, see Table 1. The estimates of

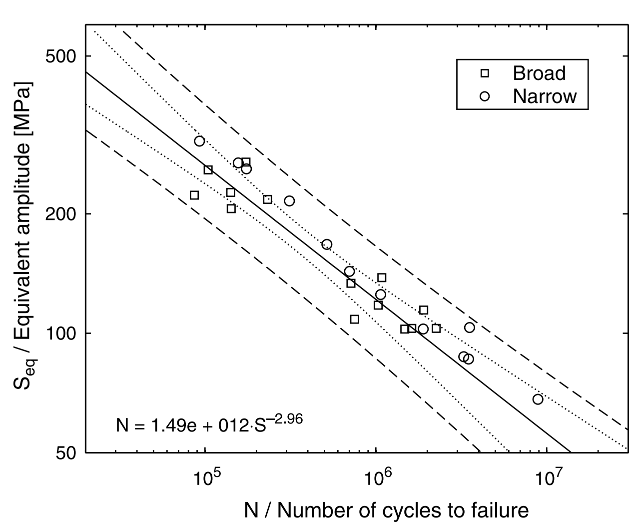

Fig. 5. *SN*-curve estimated from Broad spectrum tests, with prediction and confidence bands for Narrow spectrum.

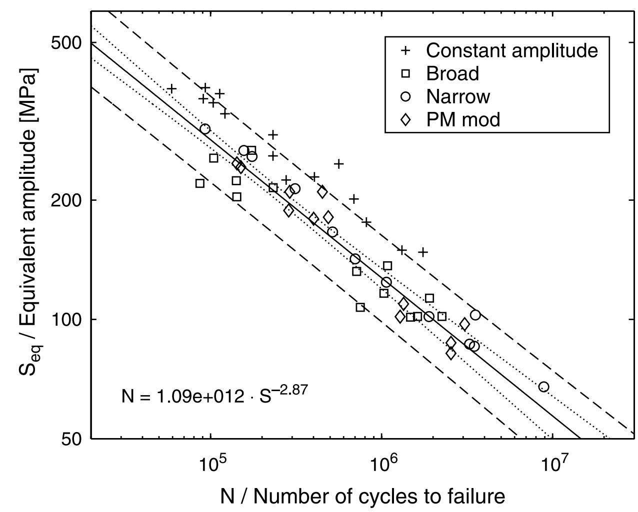

Fig. 6. *SN*-curve estimated from all variable amplitude tests, with prediction and confidence bands for constant amplitude.

the parameters a, β, and σ^2 in Eq. (7) are (asymptotically) independent. However, the parameter α is a function of both a and β, and its estimate is strongly dependent on the estimate of β. Therefore, we have instead chosen to present the estimated life for a typical constant amplitude, N_{200}, which is almost independent of β. As was observed before, constant amplitude tests predict longer lives than do VA tests. For welded structures a slope of $\beta = 3$ is often used, and we observe that this value is covered by all confidence intervals. There is a tendency that the Narrow spectrum gives longer lives and smaller scatter than the two broad band spectra, but this is not statistically significant.

In Table 2 the estimates from the different VA tests are compared with the CA test, showing no significant differences, neither in the slope β, nor in the variance σ^2 of observed log lives. The difference in N_{200} is statistically significant for all spectra, showing that the parameter α is different in the case of variable amplitude compared to constant amplitude.

4.4. Conclusions from Agerskov data

The conclusion of this case study is that constant amplitude tests predict a life twice as long as one predicted by variable amplitude tests. However, no difference in

Table 1
Estimated parameters with 95% confidence intervals

Data	Parameters		
	N_{200}	β	σ
CA	610 (509, 732)	3.07 (2.64,3.51)	0.25 (0.15,0.35)
Broad	230 (180, 294)	2.96 (2.32,3.60)	0.40 (0.24,0.57)
Narrow	330 (288, 378)	2.87 (2.60,3.15)	0.21 (0.11,0.30)
PM mod	281 (234, 336)	2.56 (2.11,3.01)	0.28 (0.15,0.41)
All VA	274 (244, 307)	2.87 (2.60,3.14)	0.34 (0.26,0.42)

The life N_{200} at constant amplitude 200 MPa is presented instead of α

Table 2
Comparison of variable amplitude and constant amplitude tests

Data	Compared parameters		
	$N_{200}/N_{200,\mathrm{CA}}$	$\beta-\beta_{\mathrm{CA}}$	$\sigma/\sigma_{\mathrm{CA}}$
Broad	0.38 (0.28,0.50)	−0.11 (−0.83,0.60)	1.60 (0.90,2.88)
Narrow	0.54 (0.44,0.66)	−0.20 (−0.67,0.27)	0.82 (0.46,1.55)
PM mod	0.46 (0.36,0.58)	−0.52 (−1.09,0.05)	1.11 (0.61,2.10)
All VA	0.45 (0.37,0.55)	−0.21 (−0.69,0.28)	1.36 (0.81,2.06)

Estimated values and 95% confidence intervals are given.

the slope β is seen. The difference is a multiplicative factor on α, which means a parallel translation of the *SN*-curve in the log-log-diagram. In terms of the relative Miner rule it would mean to set the critical damage value to 0.5 when predicting for variable amplitude loads using constant amplitude tests. With our proposed method of estimating the Wöhler curve directly from spectrum tests, one could say that this empirical correction factor is automatically included, and the original Palmgren-Miner rule can be used when predicting the life of other spectra.

It is often claimed that VA tests give smaller scatter in life, while in this case no difference is detected. The *SN*-curve estimated from one of the spectra is consistent with the observed lives from the others, hence no systematic prediction errors are detected between different variable amplitude spectra.

5. Discussion of similar proposals

The lack of validity of the traditional Palmgren-Miner rule has been observed in many engineering applications. Different approaches for adjusting the usual method with the aid of variable amplitude tests have been presented. Our interpretation of these methods is as follows:

Gassner line. Gassner [6] introduced a Wöhler diagram based on spectrum tests in the 1950s. He plotted the life as a function of the maximum load in the spectrum, i.e. illustrated the relationship

$$N = \alpha(S_{\max})^{-\beta}. \tag{21}$$

This relationship was then used for prediction of failures in service. This procedure does not include the traditional damage summation hypothesis, but only assumes that the Basquin Eq. (1) is valid.

The Gassner method excludes the constant amplitude tests and makes reference tests with the very spectrum type that is expected to appear in service. Consequently, the method is limited to situations where the expected service loads can be represented by a single spectrum that can be reproduced in laboratory. Further, it requires new reference tests for each type of spectrum.

Relative Miner rule. Based on observations that the Palmgren-Miner rule did not give good predictions, Schütz and others presented the relative Miner rule [18,16]. Our description is based on Buch [5]. The Wöhler parameters are estimated from constant amplitude tests. Then reference spectrum tests are performed in laboratory and the experimental *Miner sum* is calculated according to Eq. (2) for each spectrum test at failure

$$D^* = \frac{N_f}{\hat{\alpha}} \sum_i \nu_i S_i^{\hat{\beta}} = \frac{N_f}{N_{\mathrm{pred}}} \tag{22}$$

where N_f is the number of cycles to failure and N_{pred} is the predicted life according to the Palmgren-Miner rule. In the case of several reference tests, the geometric mean value $\overline{D^*}$ is used. Life prediction for a new spectrum is then performed by replacing the traditional

$$N_{\mathrm{pred}} = \frac{\hat{\alpha}}{\sum_k \hat{\nu}_k \hat{S}_k^{\hat{\beta}}} \quad \text{with the relative} \qquad N_{\mathrm{pred}}^{(\mathrm{relative})} = \frac{D^* \hat{\alpha}}{\sum_k \hat{\nu}_k \hat{S}_k^{\hat{\beta}}}. \tag{23}$$

This procedure is the same as predicting failure at a damage sum of D^* instead of the value 1 for the Palmgren-Miner rule

$$\sum_i \frac{n_i}{N_i} = D^* \Leftrightarrow \sum_i \frac{n_i}{D^* N_i} = 1. \tag{24}$$

Hence, changing the damage sum at failure to D^* is equivalent to keeping the damage sum at failure to 1, but changing the number of cycles to failure to $N_i^* = D^* N_i$. In the case of the Basquin curve we have

$$N_i^* = D^* N_i = D^* \alpha S_i^{-\beta} = \alpha^* S_i^{-\beta} \tag{25}$$

and the relative Miner rule implies replacing α with $\alpha^* = D^* \alpha$.

To sum up, the *relative Miner rule* is equivalent to the procedure of first estimating the exponent parameter β from constant amplitude tests, and then estimating the parameter α from spectrum reference tests.

Jarfall and Olsson. Jarfall presented a new method for airspace applications in [7]. Laboratory tests are performed with a certain reference spectrum scaled by different factors. A certain Wöhler exponent β is chosen based on previous experience and the parameter α is estimated from the tests. The Duty/Capacity-model presented by K-E Olsson [11] also contains the same idea of using spectrum reference tests for estimating α, assuming a known exponent β.

The main difference between this and our proposal is that here the exponent β is known beforehand and only the parameter α is estimated from reference laboratory tests, using the Palmgren-Miner hypothesis. Since only one parameter is estimated, the test spectrum need not be scaled to different levels, but only one level close to the actual application may be chosen.

Omerspahic. A similar approach as the actual proposal was presented in a Master thesis, and evaluated for experiments on spot-welded components

[12]. Here the parameter estimation was performed in two steps; first the exponent β was estimated based on the maximum ranges in the spectra using Eq. (21). The result could then be used to calculate a property, using the Palmgren-Miner hypothesis, that is the same as the equivalent load amplitude, Eq. (3), and then the parameter α is estimated using Eq. (4). This procedure was done for two different reference spectra and resulted in two different pairs of parameter estimates. The approach does not allow simultaneous estimation based on tests with different types of spectra. Further, the statistical uncertainties in the estimation and prediction are not considered.

The proposed method is based on the Palmgren-Miner hypothesis, and estimates both the parameters α and β of the *SN*-curve from the laboratory reference spectrum tests, and allows the use of different types of reference spectra. Both uncertainties in parameter estimation and life prediction are addressed.

The Gassner method has the advantage that it includes a minimum of model assumptions, which unfortunately also results in a corresponding lack of generality. The Gassner, the Omerspahic and the proposed method all need scaled spectrum tests in order to be able to estimate the parameters. This makes it important with a careful choice of scaled spectra, since the hypothetical linear part of the Wöhler diagram is limited both at high and at low loads. The other methods do not need scaled spectra. The relative Miner rule avoids this problem by using constant amplitude tests for estimation of the slope, and the Jarfall and Olsson methods avoid it by assuming a slope based on experience.

6. Discussion on extensions of the methodology

It is possible to extend our model to include mean stress corrections, or crack closure models. These models require a more detailed description of the load than just the load spectrum.

6.1. Mean stress correction

Several methods for mean stress correction are found in literature, where each cycle with amplitude s_a and mean s_m is transformed into a damage equivalent cycle with amplitude $s'_a = h(s_a, s_m)$ at mean zero. Hence, both the mean and the amplitude of the cycles are needed. Given that the mean stress correction is known, we can construct a load spectrum corrected for the mean stress effect by defining a spectrum with amplitudes $S_j = s'_{a,j} = h(s_{a,j}, s_{m,j})$ with frequencies ν_j according to the frequency of the cycle ($s_{a,j}$, $s_{m,j}$). Our modified load spectrum can then be used for estimation or prediction.

A simple empirical correction is a linear curve in the Haigh diagram

$$s'_a = h(s_a, s_m) = s_a + M s_m \tag{26}$$

where M is the so-called mean-stress-sensibility of the material, Schütz [17]. It has, for example, been suggested for spot welds, Rupp et al. [14], where the parameter M is found from experiments. However, it is possible to include the parameter M in our model and define the equivalent amplitude as a function of both β and M

$$S_{eq} = \sqrt[\beta]{\sum_j \nu_j h(s_{a,j}, s_{m,j})^\beta} = \sqrt[\beta]{\sum_j \nu_j (s_{a,j} + M s_{m,j})^\beta} \tag{27}$$

and estimate M together with the parameters in the *SN*-curve. The estimation is done by also including M in the minimization of Eq. (10).

6.2. Crack closure models

Crack closure models assume knowledge about the load sequence, in order to calculate an individual closure level for each cycle, see e.g. Kahlil et al. [8]. The effective cycle range is the difference between the cycle maximum and the closure level. A simplified approach for variable amplitude loads is to assume a constant closure level, see e.g. Svensson [15], that depends on the properties of the global maximum and minimum of the load. With knowledge about the cycle minima and maxima we can define an effective load spectrum with frequencies ν_j of effective ranges

$$S_j = S_{\mathrm{eff},j} = \begin{cases} S_{\max,j} - S_{cl}, & \text{if} \quad S_{\min,j} < S_{cl} \\ S_{\max,j} - S_{\min,j}, & \text{if} \quad S_{\min,j} \geq S_{cl} \end{cases} \tag{28}$$

where the closure level S_{cl} depends on the spectrum. The use of an effective spectrum for estimation would result in an effective *SN*-curve. By defining a model for the closure level, it is also possible to estimate the crack closure parameters directly from our spectrum tests.

6.3. Combining results from CA and VA tests

There is also the possibility to extend our methodology by using a combination of test results from both constant and variable amplitude. If we assume that the slope β of the *SN*-curves for CA and VA is the same, it would be possible to combine the test results to get a better estimate of β.

One possibility is to estimate the *SN*-curves for CA and VA, both at the same time. The extended model then contains two *SN*-curves, one for CA and one for VA, where the slope β is the same, but the strength α in the *SN*-curve is different, giving two parameters α_{CA} and α_{VA}. This will give a model with four parameters (α_{CA}, α_{VA}, β, σ^2). If also the uncertainty σ^2 around the *SN*-curve is different for CA and VA, it will introduce an extra parameter, giving five parameters (α_{CA}, α_{VA}, β, σ^2_{CA}, σ^2_{VA}). The maximum likelihood estimation for the extended models need to be derived, using the same technique as for the case of only three parameters (α, β, σ^2). This idea can also be generalised to deal with different types of spectra, each having a different α.

Another possibility, motivated by the two step procedure of the relative Miner rule, is to use a Bayesian approach. In the first step, the CA tests give an estimate of β and its distribution, which is then used in the second step as prior information on β, when making a Bayesian estimation of the *SN*-curve based on the VA tests.

7. Conclusions

It is well known that fatigue life predictions, based on a Wöhler curve from tests at constant amplitude, often give large systematic prediction errors for service loads at variable amplitude. Berger et al. [4] found that the Miner sum at failure was in the range of 0.01–10. Many refinements of the fatigue damage calculations have been suggested, especially mean stress corrections, and also crack closure models. Another solution is to modify the load of the experiment, e.g. test at an optimal *R*-value, test with a specific service load (Gassner line), or calibrate the predictions with a few variable amplitude tests (relative Miner rule). Our proposal lies in the second category, and the method is to test at different variable amplitude loads that are representative for the service. These tests are then the base for the predictions. The main contribution of this paper is the method for estimating the *SN*-curve from different variable amplitude load spectra, where also the statistical uncertainties of the estimated parameters and in the predicted lives are treated in detail. The statistical analysis makes it possible to distinguish a systematic model error from a possible random error.

In the example of welded specimens of Agerskov [1], it was found that when using constant amplitude tests for predicting variable amplitude loads, it gives non-conservative predictions with about a factor of two for the three different load spectra investigated. When comparing the test results, no difference could be found in the Wöhler exponent, while the position of the Wöhler curve was found to be different for constant and variable amplitude loads. When prediction life for a spectrum load based on test from other types of spectra, no systematic prediction errors could be detected. This indicates that the Palmgren-Miner hypothesis in fact works well. The discrepancy often observed is that the damage caused by a cycle with a given amplitude is different for the cases of constant and variable amplitude loads.

It is our belief that the proposed method will lead to more accurate predictions, compared to constant amplitude tests, for a wide range of materials and applications, mainly by reducing or eliminating the systematic error in the life predictions for variable amplitude loads.

Acknowledgements

The financial support from Atlas Copco RockDrills, Bombardier Transportation, Sandvik Steel, Volvo Cars, and the Swedish Foundation for Applied Mathematics is gratefully acknowledged. The authors would also like to thank the Smögen Workshop for an inspiring environment. Further, the authors are in debt to Professor Sture Holm for discussions concerning the maximum likelihood estimation.

Appendix A. Maximum likelihood estimation of parameters of *SN*-curve from variable amplitude tests

A.1. Model

We assume a *SN*-curve model with multiplicative independent log-normal errors ϵ_i

$$N_i = \alpha S_{\mathrm{eq},i}^{-\beta}\epsilon_i, \quad i = 1,\ldots,n \tag{A.1}$$

where n is the number of tests. Taking logs yield the linear model

$$\ln N_i = \ln\alpha - \beta\ln S_{\mathrm{eq},i} + \ln\epsilon_i \tag{A.2}$$

where the errors $e_i = \ln\epsilon_i$ are $N(0,\sigma^2)$ and independent. To avoid numerical problems when estimating the uncertainty of the estimates we rewrite the model by subtracting the mean of $\ln S_{\mathrm{eq},i}$

$$\ln N_i = \ln\alpha - \beta\overline{\ln S_{\mathrm{eq}}} - \beta(\ln S_{\mathrm{eq},i} - \overline{\ln S_{\mathrm{eq}}}) + e_i,$$
$$\overline{\ln S_{\mathrm{eq}}} = \frac{1}{n}\sum_{j=1}^{n}\ln S_{\mathrm{eq},j}. \tag{A3}$$

A reparametrisation

$$a = \ln\alpha - \beta\overline{\ln S_{\mathrm{eq}}}, \quad b = \beta \tag{A.4}$$

and setting $y_i = \ln N_i$ gives

$$y_i = a - b(\ln S_{\mathrm{eq},i} - \overline{\ln S_{\mathrm{eq}}}) + e_i \tag{A.5}$$

which we recognise as a regression model, however non-linear since $\ln S_{\mathrm{eq},i}$ and $\overline{\ln S_{\mathrm{eq}}}$ depend on b.

The equivalent load $S_{\mathrm{eq},i}$ is defined in Eq. (3), however it should be noted that $S_{\mathrm{eq},i}^b$ is the mean of the amplitudes to the power of b, i.e.

$$S_{\mathrm{eq},i}^b = \sum_k \nu_{i,k} S_{i,k}^b = \mathbf{E}[S_i^b]. \tag{A.6}$$

Concequently, the model can now be written as

$$y_i = a - \ln\mathbf{E}[S_i^b] + \frac{1}{n}\sum_{j=1}^{n}\ln\mathbf{E}[S_j^b] + e_i = m_i(a,b) + e_i. \tag{A.7}$$

We conclude that y_i exhibits a normal distribution with a mean function

$$m_i(a,b) = a - b(\ln S_{\text{eq},i} - \overline{\ln S_{\text{eq}}}) = a - \ln \mathbf{E}[S_i^b] + \frac{1}{n}\sum_{j=1}^{n} \ln \mathbf{E}[S_j^b] \quad (A.8)$$

and variance σ^2.

The parameters α and β of the *SN*-curve Eq. (A.1) are obtained as

$$\alpha = \exp(a + b\overline{\ln S_{\text{eq}}}) = \exp(a + \frac{1}{n}\sum_{j=1}^{n} \ln \mathbf{E}[S_j^b]) \quad (A.9)$$

$$\beta = b. \quad (A.10)$$

A.2. Maximum likelihood estimation

The observations y_i are independent and $N(m_i(a, b),\sigma^2)$. The estimates are obtained by maximising the likelihood function

$$L(a,b,\sigma;y_1,\ldots,y_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y_i - m_i(a,b))^2}{2\sigma^2}\right) \quad (A.11)$$

or equivalently by maximising the log-likelihood

$$l(a,b,\sigma;y_1,\ldots,y_n) = \ln L(a,b,\sigma;y_1,\ldots,y_n) \quad (A.12)$$

$$l(a,b,\sigma;y_1,\ldots,y_n) = -n\ln(2\pi)/2 - n\ln\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - m_i(a,b))^2 \quad (A.13)$$

By differentiating with respect to a, b, and σ, and solving the derivatives equal zero, we get

$$a = \frac{1}{n}\sum_{i=1}^{n} y_i = \bar{y}, \quad (A.14)$$

$$0 = \sum_{i=1}^{n}(y_i - m_i(a,b))\frac{\partial m_i}{\partial b}(a,b), \quad (A.15)$$

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - m_i(a,b))^2 \quad (A.16)$$

where

$$\frac{\partial m_i}{\partial b}(a,b) = -\frac{\mathbf{E}[S_i^b \ln S_i]}{\mathbf{E}[S_i^b]} + \frac{1}{n}\sum_{j=1}^{n}\frac{\mathbf{E}[S_j^b \ln S_j]}{\mathbf{E}[S_j^b]}. \quad (A.17)$$

We observe that the first two equations do not depend on σ; we can thus solve them first. Eq. (A.15) for b can not be solved explicitly, and the zero need to be found numerically. However, a better alternative is to directly maximise the likelihood by some numerical method. We observe that $l(a, b, \sigma;y_1,\ldots,y_n)$ is maximised with respect to a and b when $a = \hat{a} = \bar{y}$ and

$$g(b) = \sum_{i=1}^{n}(y_i - m_i(\hat{a},b))^2 \quad (A.18)$$

is minimised. This minimisation problem is easily solved by quasi-Newton methods, or by non-linear least squares methods. Such methods are found in most numerical optimization packages, e.g. in the optimization toolbox in Matlab.

Once the estimates $(\hat{a},\hat{b})$ of (a, b) are computed they are plugged into Eq. (A.16) to get the maximum likelihood estimate of σ^2

$$s_{\text{ML}}^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i - m_i(\hat{a},\hat{b}))^2. \quad (A.19)$$

However, this estimate is biased, and the bias corrected estimate of σ^2 is

$$s^2 = \frac{1}{n-2}\sum_{i=1}^{n}(y_i - m_i(\hat{a},\hat{b}))^2. \quad (A.20)$$

The next issue is to find the distribution of the estimates. The information matrix $\boldsymbol{I}(\boldsymbol{\theta})$ with $\boldsymbol{\theta}=(a, b, \sigma)$ is defined as $I_{ij}(\boldsymbol{\theta}) = E_{\boldsymbol{\theta}}\left[-\frac{\partial^2 l}{\partial\theta_i\partial\theta_j}\right]$. It is a well-known property of maximum likelihood estimation (see e.g. Lehmann [10, Chapter 6]) that, under some regularity conditions, the estimates are asymptotically normally distributed with variance matrix equal to the inverse of the information matrix, $\mathbf{V}[\hat{\boldsymbol{\theta}}] = \mathbf{I}(\boldsymbol{\theta})^{-1}$. The second derivatives of the log-likelihood function is

$$-\frac{\partial^2 l}{\partial a^2} = \frac{n}{\sigma^2} \quad (A.21)$$

$$-\frac{\partial^2 l}{\partial b^2} = \frac{1}{\sigma^2}\sum_{i=1}^{n}\left(\left(\frac{\partial m_i}{\partial b}(a,b)\right)^2 - \frac{\partial^2 m_i}{\partial b^2}(a,b)(y_i - m_i(a,b))\right) \quad (A.22)$$

$$-\frac{\partial^2 l}{\partial \sigma^2} = \frac{3\sum_{i=1}^{n}(y_i - m_i(a,b))^2 - n\sigma^2}{\sigma^4} \quad (A.23)$$

$$-\frac{\partial^2 l}{\partial a \partial b} = 0 \quad (A.24)$$

$$-\frac{\partial^2 l}{\partial a \partial \sigma} = \frac{2}{\sigma^3}\sum_{i=1}^{n}(y_i - m_i(a,b)) \quad (A.25)$$

$$-\frac{\partial^2 l}{\partial b \partial \sigma} = \frac{2}{\sigma^3}\sum_{i=1}^{n}(y_i - m_i(a,b))\frac{\partial m_i}{\partial b}(a,b) \quad (A.26)$$

and taking expectations using $\mathbf{E}[y_i - m_i(a,b)] = 0$ and $\mathbf{E}[(y_i - m_i(a,b))^2] = \sigma^2$ yields

$$\boldsymbol{I}(\boldsymbol{\theta}) = \frac{1}{\sigma^2}\begin{pmatrix} n & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & 2n \end{pmatrix} \tag{A.27}$$

where

$$q = \sum_{i=1}^{n}\left(\frac{\partial m_i}{\partial b}(a,b)\right)^2 = \sum_{i=1}^{n}(c_i-\bar{c})^2, \quad c_i = \frac{\mathbf{E}[S_i^b \ln S_i]}{\mathbf{E}[S_i^b]}, \quad \bar{c} = \frac{1}{n}\sum_{i=1}^{n} c_i. \tag{A.28}$$

Inverting the information matrix gives the asymptotic variance matrix for $\hat{\boldsymbol{\theta}} = (\hat{a},\hat{b},\hat{\sigma})$

$$\mathbf{V}[\hat{\boldsymbol{\theta}}] = \frac{\sigma^2}{n}\begin{pmatrix} 1 & 0 & 0 \\ 0 & n/q & 0 \\ 0 & 0 & 1/2 \end{pmatrix}. \tag{A.29}$$

For s^2 we can derive a more exact result. Since Eq. (A.19) is a sum of residuals that are normally distributed (at least asymptotically) we get a χ^2-distribution

$$(n-2)\frac{s^2}{\sigma^2} \in \chi^2(n-2). \tag{A.30}$$

By assuming what is known asymptotically, i.e. independence of the estimates and that $\hat{a} \in N(a,\sigma^2/n)$, $\hat{b} \in N(b,\sigma^2/q)$, and $(n-2)s^2/\sigma^2 \in \chi^2(n-2)$, we get the $1-p$ confidence intervals for the parameters

$$I_a = (\hat{a} - t_{1-p/2}(n-2)\cdot s/\sqrt{n},\ \ \hat{a} + t_{1-p/2}(n-2)\cdot s/\sqrt{n}) \tag{A.31}$$

$$I_b = (\hat{b} - t_{1-p/2}(n-2)\cdot s/\sqrt{q},\ \ \hat{b} + t_{1-p/2}(n-2)\cdot s/\sqrt{q}), \tag{A.32}$$

$$I_\sigma = \left(s\sqrt{\frac{n-2}{\chi^2_{1-p/2}(n-2)}},\ \ s\sqrt{\frac{n-2}{\chi^2_{p/2}(n-2)}}\right). \tag{A.33}$$

The confidence interval for $\alpha = \exp\left(a + \frac{1}{n}\sum_{j=1}^{n}\ln \mathbf{E}[S_j^b]\right)$ is

$$I_\alpha = \exp\left(\ln\hat{\alpha} \pm t_{1-p/2}(n-2)\cdot s\sqrt{\frac{1}{n}+\frac{\bar{c}^2}{q}}\right). \tag{A.34}$$

A.3. Life prediction and prediction bands

From our estimated *SN*-curve we want to predict life for a given VA load with load spectrum $\hat{S}$. The prediction is calculated as

$$\hat{N} = \frac{\hat{\alpha}}{\hat{S}_{eq}^{\hat{\beta}}} = \frac{\hat{\alpha}}{\mathbf{E}[\hat{S}^{\hat{\beta}}]} \tag{A.35}$$

which is identical to the prediction obtained using the Palmgren-Miner sum. Now we will derive the confidence intervals for the median life and the prediction bands. Taking logs give

$$\ln\hat{N} = \ln\hat{\alpha} - \ln \mathbf{E}[\hat{S}^{\hat{\beta}}] \tag{A.36}$$

and rewriting using the estimates $\hat{a}$ and $\hat{b}$ yields

$$\hat{y} = \hat{a} - \ln \mathbf{E}[\hat{S}^{\hat{b}}] + \frac{1}{n}\sum_{i=1}^{n}\ln \mathbf{E}[S_i^{\hat{b}}] = \hat{a} - h(\hat{b}). \tag{A.37}$$

The variance of the estimated median life can be computed approximately using Gauss approximation formula

$$\mathbf{V}[\hat{y}] = \mathbf{V}[\hat{a}] + \mathbf{V}[h(\hat{b})] \approx \mathbf{V}[\hat{a}] + c_{\hat{b}}^2\mathbf{V}[\hat{b}] \tag{A.38}$$

with

$$c_{\hat{b}} = \frac{dh}{db}(\hat{b}) = \frac{\mathbf{E}[\hat{S}^{\hat{b}}\ln\hat{S}]}{\mathbf{E}[\hat{S}^{\hat{b}}]} - \frac{1}{n}\sum_{i=1}^{n}\frac{\mathbf{E}[S_i^{\hat{b}}\ln S_i]}{\mathbf{E}[S_i^{\hat{b}}]} = \hat{c} - \bar{c}. \tag{A.39}$$

Inserting the variances of $\hat{a}$ and $\hat{b}$ gives

$$\mathbf{V}[\hat{y}] \approx \sigma^2\left(\frac{1}{n} + \frac{(\hat{c}-\bar{c})^2}{q}\right). \tag{A.40}$$

A new observation $\tilde{y} = \ln\tilde{N}$ is attached with an independent random error $\tilde{e}$, which means that for the prediction bands, this error should also be added. The prediction error is

$$\Delta = \tilde{y} - \hat{y} = m(a,b) + \tilde{e} - \hat{y} \tag{A.41}$$

where $m(a,b)$ is the true *SN*-curve. The variance of Δ is

$$\mathbf{V}[\Delta] = \mathbf{V}[\hat{y}] + \mathbf{V}[\tilde{e}] \approx \sigma^2\left(\frac{1}{n} + \frac{(\hat{c}-\bar{c})^2}{q} + 1\right). \tag{A.42}$$

Inserting the estimates give the $1-p$ confidence interval for the median life

$$I_{\hat{N}} = \left(\exp\left(\ln\hat{N} \pm t_{1-p/2}(n-2)\cdot s\sqrt{\frac{1}{n}+\frac{(\hat{c}-\bar{c})^2}{q}}\right)\right) \tag{A.43}$$

and the $1-p$ prediction interval for a new test

$$I_{\tilde{N}} = \left(\exp\left(\ln\hat{N} \pm t_{1-p/2}(n-2)\cdot s\sqrt{\frac{1}{n}+\frac{(\hat{c}-\bar{c})^2}{q}+1}\right)\right). \tag{A.44}$$

In order to detect a systematic model error, we are also interested in examining the mean of several prediction errors

$$\delta = \frac{1}{r}\sum_{k=1}^{r}\Delta_k, \tag{A.45}$$

$$\Delta_k = \tilde{y}_k - \hat{y}_k = m_k(a,b) + \tilde{e}_k - \hat{y}_k, \tag{A.46}$$

$$\hat{y}_k = \hat{a} - \ln \mathbf{E}[\hat{S}_k^{\hat{b}}] + \frac{1}{n}\sum_{i=1}^{n}\ln \mathbf{E}[S_i^{\hat{b}}] = \hat{a} - h_k(\hat{b}). \tag{A.47}$$

Since all predictions are based on the same estimate of the *SN*-curve, the prediction errors are correlated, and the variance of δ is

$$\mathbf{V}[\delta] = \frac{1}{r^2}\sum_{k=1}^{r}\sum_{l=1}^{r} C[\Delta_k, \Delta_l] \tag{A.48}$$

where $\mathbf{C}[\Delta_k, \Delta_l]$ can be approximated as

$$\mathbf{C}[\Delta_k, \Delta_l] \overset{k=l}{=} \mathbf{V}[\Delta_k] = \mathbf{V}[\hat{a}] + \mathbf{V}[h_k(\hat{b})] + \mathbf{V}[\tilde{e}_k] \tag{A.49}$$

$$\approx \mathbf{V}[\hat{a}] + (\hat{c}_k - \bar{c})^2\mathbf{V}[\hat{b}] + \mathbf{V}[\tilde{e}_k], \tag{A.50}$$

$$= \sigma^2\left(\frac{1}{n} + \frac{(\hat{c}_k - \bar{c})^2}{q} + 1\right) \tag{A.51}$$

$$\mathbf{C}[\Delta_k, \Delta_l] \overset{k\neq l}{=} \mathbf{C}[\hat{a} + h_k(\hat{b}) + \tilde{e}_k, \hat{a} + h_l(\hat{b}) + \tilde{e}_l] \tag{A.52}$$

$$= \mathbf{V}[\hat{a}] + \mathbf{C}[h_k(\hat{b}), h_l(\hat{b})] \tag{A.53}$$

$$\approx \mathbf{V}[\hat{a}] + \mathbf{C}[(\hat{c}_k - \bar{c})\hat{b}, (\hat{c}_l - \bar{c})\hat{b}] \tag{A.54}$$

$$= \mathbf{V}[\hat{a}] + (\hat{c}_k - \bar{c})(\hat{c}_l - \bar{c})\mathbf{V}[\hat{b}] \tag{A.55}$$

$$= \sigma^2\left(\frac{1}{n} + \frac{(\hat{c}_k - \bar{c})(\hat{c}_l - \bar{c})}{q}\right) \tag{A.56}$$

where we used the Taylor expansion of $h_k(\hat{b})$

$$h_k(\hat{b}) \approx h_k(\mathbf{E}[\hat{b}]) + \left.\frac{\mathrm{d}h_k}{\mathrm{d}\hat{b}}\right|_{\hat{b}=\mathbf{E}[\hat{b}]}(\hat{b} - \mathbf{E}[\hat{b}]) \tag{A.57}$$

where the derivative defines the coefficients $\hat{c}_k$

$$\frac{\mathrm{d}h_k}{\mathrm{d}\hat{b}} = \hat{c}_k - \bar{c} \quad \text{with} \quad \hat{c}_k = \frac{\mathbf{E}[\hat{S}_k^{\hat{b}} \ln \hat{S}_k]}{\mathbf{E}[\hat{S}_k^{\hat{b}}]} \tag{A.58}$$

(compare Eq. (A.39)). Finally, we get

$$\mathbf{V}[\delta] = \frac{1}{r^2}\sigma^2\left(r^2\frac{1}{n} + \sum_{k=1}^{r}\sum_{l=1}^{r}\frac{(\hat{c}_k - \bar{c})(\hat{c}_l - \bar{c})}{q} + r\right) \tag{A.59}$$

$$\mathbf{V}[\delta] = \sigma^2\left(\frac{1}{n} + \frac{1}{r^2}\sum_{k=1}^{r}\sum_{l=1}^{r}\frac{(\hat{c}_k - \bar{c})(\hat{c}_l - \bar{c})}{q} + \frac{1}{r}\right). \tag{A.60}$$

The confidence interval for the systematic model error δ is

$$I_\delta = \left(\delta \pm t_{1-p/2}(n-2)\cdot s\sqrt{\frac{1}{n} + \frac{\tilde{c}}{q} + \frac{1}{r}}\right) \tag{A.61}$$

with

$$\tilde{c} = \frac{1}{r^2}\sum_{k=1}^{r}\sum_{l=1}^{r}(\hat{c}_k - \bar{c})(\hat{c}_l - \bar{c}). \tag{A.62}$$

For the special case where all predictions are made for the same spectrum, we have $\hat{c} = \hat{c}_k = \hat{c}_l$, giving

$$\mathbf{V}[\delta] = \sigma^2\left(\frac{1}{n} + \frac{(\hat{c} - \bar{c})^2}{q} + \frac{1}{r}\right) \tag{A.63}$$

where we clearly can see, by comparing Eq. (A.42), that the uncertainty due to the estimation remains unchanged while the uncertainty due to the inherent errors $\tilde{e}_k$ decreases with the number of predictions, from σ^2 to σ^2/r.

The systematic model error is best studied using the relative prediction error

$$\hat{N}_{\mathrm{rel}} = \left(\prod_{k=1}^{r}\frac{\tilde{N}_k}{\hat{N}_k}\right)^{1/r} = \exp\left(\frac{1}{r}\sum_{k=1}^{r}\ln\tilde{N}_k - \ln\hat{N}_k\right) = e^{\delta} \tag{A.64}$$

based on the geometric mean of r relative prediction errors, $\tilde{N}_k$ being the observed life and $\hat{N}_k$ the prediction. Its $1-p$ confidence interval is

$$I_{N_{\mathrm{rel}}} = \exp\left(\ln\hat{N}_{\mathrm{rel}} \pm t_{1-p/2}(n-2)\cdot s\sqrt{\frac{1}{n} + \frac{\tilde{c}}{q} + \frac{1}{r}}\right). \tag{A.65}$$

References

[1] Agerskov H. Fatigue in steel structures under random loading. J Construct Steel Res 2000;53:283–305.

[2] Agerskov H, Ibsø JB. Fatigue life of plate elements with welded transverse attatchments subjected to stochastic loading. In: Blom AK, editor. Fatigue under spectrum loading and in corrosive environments. West Midlands, UK: EMAS Publishers; 1993. p. 41–60.

[3] Basquin OH. The exponential law of endurance tests. Am Soc Testing Mater 1910;10:625–30.

[4] Berger C, Eulitz K-G, Hauler P, Kotte K-L, Naundorf H, Schuetz W, et al. Betriebsfestigkeit in Germany—an overview. Int J Fatigue 2002; 24:603–25.

[5] Buch A. Prediction of the comparative fatigue performance for realistic loading distributions. Prog Aerospace Sci 1997;33:391–430.

[6] Gassner E. Betriebsfestigkeit, eine Bemessungsgrundlage für Konstruktionsteile mit statistisch wechselnden Betriebsbeanspruchungen. Konstruktion 1954;6(3):97–104.

[7] Jarfall L. Prediction of fatigue life and crack growth using spectrum test data, SAAB-SCANIA Report TKH R-3379, Linköping, Sweden; 1985.

[8] Kahlil M, DuQuesnay DL, Topper TH. Prediction of crack-opening stress levels for service loading spectra. In: Braun AA, McKeighan PC, Niclson MA, Lohr PR, editors. Applications of automation technology in fatigue and fracture testing analysis, ASTM STP 1411. West Conshohocken, PA: American Society for Testing and Materials; 2002.
[9] Miner MA. Cumulative damage in fatigue. J Appl Mech 1945;12: A159–A64.
[10] Lehmann EL. Theory of point estimation. New York: Wiley; 1983.
[11] Olsson K-E. Fatigue reliability prediction. Scand J Metall 1989;18: 176–80.
[12] Omerspahic E. Constant and variable amplitude fatigue testing and analysis of spot welded thin sheet structures. Masters Thesis, Department of Solid Mechanics, Linköping University of Technology; 1999.
[13] Palmgren A. Die Lebensdauer von Kugellagern. Zeitschrift des Vereins Deutscher Ingenieure 1924;68:339–41.
[14] Rupp A, Störzel K, Grubisic V. Computer aided dimensioning of spot-welded automotive structures. SAE Technical Paper 950711; 1995.
[15] Svensson T. Crack closure at variable amplitude fatigue—a simpified approach. In: Blom AF, editor. Proceedings of the eighth interational fatigue congress. UK. Engineering Materials Advisory Services Ltd; 2002.
[16] Schütz D, Lowak H. Zur Verwendung von Bemessungsunterlagen aus Versuchen mit betriebsähnlichen Lastfolgen zur Lebensdauerabschätzung. LBF-Bericht Nr. FB-109; 1976.
[17] Schütz W. Über die Beziehung zwischen der Lebendauer bei konstanter und bei veränderlicher Beanspruchsamplitude und ihre Anwendbarkeit auf die Bemessung von Flugzeugbauteilen. Z.f. Flugwissenschaften 1967;15:407–19.
[18] Schütz W, Zenner H. Schadensakkumulationshypotesen zur Lebensdauervorhersage bei schwingender Beanspruchung. Teil 2 - Ein Kritischer Überblick. Z.f. Werkstoffmechanik 1973;4:97–102.